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Ergodicity for generalized Kawasaki dynamics

G Kondrat†, S Peszat‡ and B Zegarlinski

Department of Mathematics, Imperial College, London SW7 2BZ, UK

E-mail: gkon@ift.uni.wroc.pl, napeszat@cyf-kr.edu.pl and b.zegarlinski@ic.ac.uk

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Abstract. We give a necessary and sufficient condition for a Gibbs measure μ on the product space $\Omega = (S^1)^{\mathbb{Z}^d}$ to satisfy the spectral gap or the logarithmic Sobolev inequality with the following quadratic form:

$$\mu\mathcal{K}(f) \equiv \int \sum_{k\in\mathbb{Z}^d} \left(\sum_{j\in k+Y} a_{j-k} \nabla_j f \right)^2 \mathrm{d}\mu \qquad f \in C_0^\infty(\Omega)$$

where Y is a finite set and a_l are integers. As a consequence we prove that the generalized Kawasaki dynamics decays exponentially to equilibrium in the supremum norm in a strong mixing region.

1. Introduction

It is well known that the Kawasaki dynamics for discrete spin systems exhibits a different behaviour from the Glauber dynamics and even at high temperatures the decay to equilibrium is very slow (cf [De, BZ1, BZ2, JLQY, LY, CM]). Naturally one can ask what happens in the case of generalized Kawasaki dynamics [ZZ] with a continuous single spin space, where the generator $\mathcal{L}$ is formally given as follows:

$$\mu(f(-\mathcal{L}f)) = \sum_{|i-j|=1} \mu|\nabla_i f - \nabla_j f|^2 \tag{1.1}$$

with μ being an equilibrium measure and the summation on the right-hand side is extending over the nearest-neighbour sites of an integer lattice. As indicated in [ZZ] such a model is of interest for describing a ferroelectric gas.

We show that, in contrast to the discrete case, if the single spin space is given by a unit circle, due to an additional 'gauge' symmetry, at high temperatures the generalized Kawasaki dynamics is hypercontractive. We also show that such a dynamics has the property of a finite speed of propagation of information, that is it can be strongly approximated by finite-dimensional dynamics. This together with the hypercontractivity property implies a strong exponential decay to equilibrium in the supremum norm.

It is now well known that the above-mentioned features are present in the models of dynamics with a generator defined by the following standard Dirichlet form:

$$\mu\mathcal{D}(f) \equiv \int \sum_{k\in\mathbb{Z}^d} |\nabla_k f|^2 \,\mathrm{d}\mu \qquad f \in C_0^\infty(\Omega) \tag{1.2}$$

† On leave from: Institute of Theoretical Physics, University of Wroclaw, Pl. Maxa Borna 9, 50-204 Wroclaw, Poland.
‡ On leave from: Institute of Mathematics, Polish Academy of Sciences, Św. Tomasza 30/7, 31-027 Kraków, Poland.

in the mixing region (see, e.g., [HS2, GZ1, GZ2]) for some earlier study of the models with continuous symmetry see [F, HS1] (for existence, uniqueness and some regularity properties of the corresponding processes) and [W] (including ergodicity in the uniform norm but at very high temperatures). Despite this similarity, even in our simple setting with a single spin space given by a circle, these two dynamics can by no means be considered to be equivalent. To indicate an example we mention that (by a simple choice of trial functions), one can easily see that the corresponding quadratic forms (1.1) and (1.2) are not equivalent. (Although naturally the latter one multiplied by a positive constant dominates the former.) One could also have a different critical behaviour of both dynamics.

In the special case of a rotator system with spins taking values in a circle, we show that there is a transformation of a potential which allows one to transform some 'gauge'-invariant dynamics corresponding to a non-diagonal quadratic form (such as the generalized Kawasaki dynamics) to one given in (1.2), but with a properly transformed measure. In this restricted sense one can talk about a correspondence between two dynamics related to quadratic forms having an *a priori* different form.

The organization of the paper is as follows. After a preliminary section 2, we consider in section 3 the spin systems with a single spin space given by the unit circle and a smooth finite-range potential. For such systems we formulate a necessary and sufficient condition for a spectral gap and logarithmic Sobolev inequality to be true with some general class of Dirichlet forms, which we will call 'the square of the field forms'. The proof of this result based on an appropriate change of integration variables and a mixing property for a transformed potential is given in section 4. Section 5 contains a general example of a system with a small potential for which the required conditions are satisfied. In section 6 we discuss the construction of a Markov semigroup with generator corresponding to a general square of the field form. Finally, in section 7 we explain how to apply our general results to prove the exponential decay to equilibrium in the uniform norm for the generalized Kawasaki dynamics.

2. Preliminaries

Let $\mathbb{Z}^d$ be the d-dimensional integer lattice with the norm $|k| = \max_{1\leqslant i\leqslant d} |k^i|$. We write $k \sim j$ iff $|k-j| = 1$. We use $\mathcal{F}$ to denote the set of all non-empty $\Lambda \subset \mathbb{Z}^d$ with the cardinality $|\Lambda| < \infty$.

As a single spin space we consider the unit circle S^1, and our configuration space is the space $\Omega = (S^1)^{\mathbb{Z}^d}$ endowed with the product topology.

Given a non-empty $\Lambda \subseteq \mathbb{Z}^d$ we denote by $B_\Lambda(\Omega)$, $C_\Lambda(\Omega)$ and $C_\Lambda^\infty(\Omega)$ the spaces of bounded measurable, continuous and infinitely differentiable real-valued functions on Ω depending only on the variables ω_k, $k \in \Lambda$. We say that a function is *local* iff it belongs to $B_\Lambda(\Omega)$ for some $\Lambda \in \mathcal{F}$. We use $B_0(\Omega)$, $C_0(\Omega)$ and $C_0^\infty(\Omega)$ to denote the classes of bounded measurable, continuous and infinitely differentiable local functions on Ω. For a bounded function f on Ω, we denote by $\|f\|_{\mathrm{u}}$ the supremum norm of f. Let $\Lambda \subseteq \mathbb{Z}^d$, and let $\eta, \omega \in \Omega$. We denote by $\eta \bullet_\Lambda \omega$ the element of Ω determined by $(\eta \bullet_\Lambda \omega)_k = \eta_k$, $k \in \Lambda$ and $(\eta \bullet_\Lambda \omega)_k = \omega_k$, $k \notin \Lambda$. Given $\Lambda \subseteq \mathbb{Z}^d$, $f : \Omega \to \mathbb{R}$, and $\omega \in \Omega$ we denote by $f_\Lambda(\cdot|\omega)$ the function $f_\Lambda(\eta|\omega) = f(\eta \bullet_\Lambda \omega)$, $\eta \in \Omega$. For a (Borel) probability measure μ on Ω we use the following notation for the corresponding expectation:

$$\mu f = \int_\Omega f(\omega)\mu(\mathrm{d}\omega).$$

Let $C_0^\infty(\Omega)^2 \ni (f, g) \mapsto \mathcal{K}(f, g) \in C_0^\infty(\Omega)$ be a non-negative definite quadratic form which vanishes if f or g is a constant function. We set $\mathcal{K}(f) \equiv \mathcal{K}(f, f)$.

Definition 2.1. A probability measure μ on Ω satisfies the *spectral gap inequality with respect to $\mathcal{K}$*, in short $\mu \in \mathbf{SG}(\mathcal{K})$, if there is a constant $C < \infty$ such that

$$\mu(f - \mu f)^2 \leqslant C\mu\mathcal{K}(f) \qquad \text{for every } f \in C_0^\infty(\Omega).$$

We say that μ satisfies the *logarithmic Sobolev inequality with respect to $\mathcal{K}$*, in short $\mu \in \mathbf{LS}(\mathcal{K})$, if there is a constant $C < \infty$ such that

$$\mu f^2 \log \frac{f^2}{\mu f^2} \leqslant C\mu\mathcal{K}(f) \qquad \text{for every } f \in C_0^\infty(\Omega).$$

Remark 2.1. It is well known (see, e.g., [S]) that if $\mu \in \mathbf{LS}(\mathcal{K})$, then $\mu \in \mathbf{SG}(\mathcal{K})$.

In the present paper we denote by ν the normalized Lebesgue measure on S^1, and by μ_0 the corresponding product measure on Ω. For $\Lambda \subseteq \mathbb{Z}^d$, $\omega \in \Omega$ and $f \in B(\Omega)$ we set

$$\langle f\rangle_\Lambda(\omega) = \mu_0 f_\Lambda(\cdot|\omega) \qquad \text{and} \qquad \partial_\Lambda f(\omega) = \langle f\rangle_\Lambda(\omega) - f(\omega).$$

If $\Lambda = \{k\}$, then we will write ∂_k instead of $\partial_{\{k\}}$. Finally, by ∇_k we denote the gradient operator with respect to the kth variable.

A *potential* is by definition a family $\Phi \equiv \{\Phi_X : X \in \mathcal{F}\}$ of functions $\Phi_X \in C_X(\Omega)$ such that

$$\|\Phi\| \equiv \sup_{i\in\mathbb{Z}^d} \sum_{X\in\mathcal{F}:\ X\ni i} \|\Phi_X\|_{\mathrm{u}} < \infty.$$

The corresponding local energy functional is defined by

$$U_\Lambda = -\sum_{X\in\mathcal{F}:\ X\cap\Lambda\neq\emptyset} \Phi_X \qquad \Lambda \in \mathcal{F}.$$

By $\mathcal{E}(\Phi)$ we denote the *local specification* corresponding to Φ, that is the following family of operators

$$\mathbb{E}_\Lambda f = \frac{\left\langle f \exp\{-U_\Lambda\}\right\rangle_\Lambda}{\left\langle \exp\{-U_\Lambda\}\right\rangle_\Lambda} \qquad f \in B(\Omega) \quad \Lambda \in \mathcal{F}.$$

If $\Lambda = \{k\}$ for some point $k \in \mathbb{Z}^d$, we simplify the notation writing $U_k \equiv U_{\{k\}}$ and $\mathbb{E}_k \equiv \mathbb{E}_{\{k\}}$. We say that a probability measure μ on Ω is a *Gibbs measure for $\mathcal{E}(\Phi)$* iff

$$\mu\mathbb{E}_\Lambda f = \mu f \qquad \text{for all } \Lambda \in \mathcal{F} \quad \text{and} \quad f \in \mathcal{C}_0(\Omega).$$

We denote by $\mathcal{G}(\Phi)$ the set of all Gibbs measures for $\mathcal{E}(\Phi)$.

Remark 2.2. Note that as Ω is a compact Polish space and the local specification maps the set of continuous functions into itself, $\mathcal{G}(\Phi) \neq \emptyset$ for any potential on Ω.

We say that a potential Φ has *finite range* if there is an $R \in \mathbb{Z}_+$ such that $\Phi_X \equiv 0$ for all X with $\operatorname{diam} X \geqslant R$.

Let us denote by Λ_n the cube $[-n, n]^d \cap \mathbb{Z}^d$. For a potential Φ and a set $\Gamma \in \mathcal{F}$ such that $0 \in \Gamma$, we introduce a potential $\Phi^{(n)}$ with a cut-off as follows:

$$\Phi_X^{(n)} = \begin{cases} \Phi_X & \text{if} \quad X + \Gamma \subseteq \Lambda_n \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$U^{(n)} = -\sum_{X\in\mathcal{F}} \Phi_X^{(n)}.$$

Then, as $\Phi_X^{(n)} \equiv 0$ if $X \not\subseteq \Lambda_n$,

$$\mu^{(n)}(\mathrm{d}\omega) = \frac{\exp\{-U^{(n)}(\omega)\}}{\left\langle \exp\{-U^{(n)}\}\right\rangle_{\Lambda_n}} \mu_0(\mathrm{d}\omega)$$

is the unique Gibbs measure for $\mathcal{E}(\Phi^{(n)})$. The following lemma will be useful in the next section.

Lemma 2.1. *Let Φ be a finite-range potential, let $\Gamma \in \mathcal{F}$, $0 \in \Gamma$. Then there is a subsequence $\{n_j\}$ and a Gibbs measure $\mu \in \mathcal{G}(\Phi)$ such that*

$$\lim_{j\to\infty} \mu^{(n_j)} f = \mu f \qquad \text{for every } f \in C(\Omega). \tag{2.1}$$

Under our assumptions, the proof follows from the fact that for any given finite set $X \in \mathcal{F}$ there is an N such that for all $n > N$ one has

$$\mu^{(n)}\mathbb{E}_X f = \mu^{(n)} f$$

and one can choose a convergent subsequence to a Gibbs measure.

We use $\mathcal{G}_\Gamma(\Phi)$ to denote the class of all Gibbs measures $\mu \in \mathcal{G}(\Phi)$ such that (2.1) holds true for some sequence $\{n_j\}$. Lemma 2.1 ensures that $\mathcal{G}_\Gamma(\Phi) \neq \emptyset$.

3. Spectral gap and logarithmic Sobolev inequality for non-diagonal forms

Let $Y \in \mathcal{F}$, $Y \neq \emptyset$ and $\boldsymbol{a} = (a_i)_{i\in Y} \in (\mathbb{Z}\setminus\{0\})^Y$ be such that

$$0 \in Y \qquad 0 \notin \text{convex hull of } (Y\setminus\{0\}) \qquad \text{and} \qquad a_0 \in \{-1, 1\}. \tag{3.1}$$

Later on we will use θ to denote a pair $(Y, \boldsymbol{a})$ satisfying (3.1). Let

$$\mathcal{K}_\theta(f)(\omega) = \sum_{k\in\mathbb{Z}^d}\left(\sum_{j\in k+Y} a_{j-k}\nabla_j f(\omega)\right)^2 \qquad f \in C_0^\infty(\Omega). \tag{3.2}$$

In our considerations an important role is played by the following transformation of variables $\xi_\theta : \Omega \to \Omega$:

$$\left(\xi_\theta(\omega)\right)_j = \sum_{k\in j-Y} a_{j-k}\omega_k \qquad j \in \mathbb{Z}^d \quad \omega \in \Omega.$$

For $X \in \mathcal{F}$ we set

$$\mathcal{A}(X) = \{\tilde{X} \in \mathcal{F} : \ \tilde{X} - Y = X\}.$$

Given a potential Φ we introduce a transformed potential $\Phi^\theta \equiv \left\{\Phi_X^\theta : \ X \in \mathcal{F}\right\}$ as follows:

$$\Phi_X^\theta = \begin{cases} 0 & \text{if} \quad \mathcal{A}(X) = \emptyset \\ \displaystyle\sum_{\tilde{X}\in\mathcal{A}(X)} \Phi_{\tilde{X}} \circ \xi_\theta & \text{if} \quad \mathcal{A}(X) \neq \emptyset. \end{cases} \tag{3.3}$$

Let us denote by $\mathcal{D}$ the following (diagonal) square of the field

$$\mathcal{D}(f)(\omega) = \sum_{k\in\mathbb{Z}^d} |\nabla_k f(\omega)|^2 \qquad f \in C_0^\infty(\Omega).$$

We will prove the following equivalence theorem

Theorem 3.1. *Suppose that Φ is a finite-range potential such that there is a unique Gibbs measure $\mu \in \mathcal{G}(\Phi)$. Let Φ^θ be the corresponding transformed potential given by* (3.3). *Then:*

(a) $\mu \in \mathbf{SG}(\mathcal{K}_\theta)$ if and only if for any $\tilde{\mu} \in \mathcal{G}_{-Y}(\Phi^\theta)$ one has $\tilde{\mu} \in \mathbf{SG}(\mathcal{D})$.
(b) $\mu \in \mathbf{LS}(\mathcal{K}_\theta)$ if and only if for any $\tilde{\mu} \in \mathcal{G}_{-Y}(\Phi^\theta)$ one has $\tilde{\mu} \in \mathbf{LS}(\mathcal{D})$.

4. Proof of theorem 3.1

Let $\mathcal{E}(\Phi^\theta) = \{\mathbb{E}^\theta_\Lambda : \Lambda \in \mathcal{F}\}$ be the local specification corresponding to Φ^θ, that is

$$\mathbb{E}^\theta_\Lambda f = \frac{\langle f \exp\{-U^\theta_\Lambda\}\rangle_\Lambda}{\langle \exp\{-U^\theta_\Lambda\}\rangle_\Lambda} \qquad \text{where} \quad U^\theta_\Lambda = -\sum_{X\in\mathcal{F}:X\cap\Lambda\neq\emptyset} \Phi^\theta_X.$$

Since

$$\sum_{X\in\mathcal{F}:X\cap\Lambda\neq\emptyset} \Phi^\theta_X = \sum_{\tilde{X}\in\mathcal{F}:(\tilde{X}-Y)\cap\Lambda\neq\emptyset} \Phi_{\tilde{X}}\circ\xi_\theta = \sum_{\tilde{X}\in\mathcal{F}:\tilde{X}\cap(\Lambda+Y)\neq\emptyset} \Phi_{\tilde{X}}\circ\xi_\theta$$

we have

$$U^\theta_\Lambda = U_{\Lambda+Y}\circ\xi_\theta \qquad \Lambda\in\mathcal{F} \tag{4.1}$$

and consequently

$$\mathbb{E}^\theta_\Lambda f = \frac{\langle f \exp\{-U_{\Lambda+Y}\circ\xi_\theta\}\rangle_\Lambda}{\langle \exp\{-U_{\Lambda+Y}\circ\xi_\theta\}\rangle_\Lambda} \qquad f\in C_0^\infty(\Omega) \quad \Lambda\in\mathcal{F}.$$

Note that if Φ has a finite range, the same is true for Φ^θ.

Lemma 4.1. *Assume (3.1). Then for all cubes $\Lambda_n = [-n,n]^d\cap\mathbb{Z}^d$ and $\Lambda_l = [-l,l]^d\cap\mathbb{Z}^d$ satisfying $\Lambda_l - Y\subseteq\Lambda_n$, and for every $f\in C_{\Lambda_l}(\Omega)$ one has*

$$\langle f\circ\xi_\theta\rangle_{\Lambda_n} = \langle f\rangle_{\Lambda_n} = \langle f\rangle_{\Lambda_l}.$$

Proof. Let Λ_n, Λ_l be such that $\Lambda_l - Y\subseteq\Lambda_n$. The proof will be completed as soon as we can show that the following transformation of variables:

$$\eta_k = \begin{cases} \omega_k & \text{if} \quad k\in\Lambda_n\setminus(\Lambda_l - Y) \\ (\xi_\theta(\omega))_k & \text{if} \quad k\in\Lambda_l - Y \end{cases}$$

preserves the measure μ_0 on $\Omega_n = (S^1)^{\Lambda_n}$. To this end we introduce a lexicographic order $\{k_i\}$, $i = 1,\ldots,|\Lambda_n|$ in Λ_n satisfying

$$\{k_i : i = 1,\ldots,|\Lambda_n\setminus(\Lambda_l - Y)|\} = \Lambda_n\setminus(\Lambda_l - Y)$$

and for $i > |\Lambda_n\setminus(\Lambda_l - Y)|$,

$$(k_i - Y)\cap(\Lambda_l - Y) = \{k_r : |\Lambda_n\setminus(\Lambda_l - Y)| < r \leqslant i\}.$$

The existence of such an order is guaranteed by (3.1). Now consider the Jacobian matrix

$$A = \left\{\frac{\partial\eta_{k_i}}{\partial\omega_{k_j}}\right\} \qquad i, j = 1,\ldots,|\Lambda_n|.$$

Clearly, A is an upper-triangular matrix. Since $a_0\in\{-1,1\}$, the elements on its diagonal are from $\{-1,1\}$. Thus $|\det A| = 1$, which completes the proof. □

Lemma 4.2. *Assume that the hypothesis of theorem 3.1 is fulfilled. Let $\tilde{\mu}\in\mathcal{G}_{-Y}(\Phi^\theta)$. Then $\tilde{\mu} f\circ\xi_\theta = \mu f$ for every $f\in C_0(\Omega)$.*

Proof. Let $\tilde{\mu} \in \mathcal{G}_{-Y}(\Phi^{\theta})$. Then there is a sequence $\{n_j\}$ such that

$$\tilde{\mu} f = \lim_{j\to\infty} \frac{\left\langle f \exp\{-\tilde{U}^{(n_j)}\}\right\rangle_{\Lambda_{n_j}}}{\left\langle \exp\{-\tilde{U}^{(n_j)}\}\right\rangle_{\Lambda_{n_j}}} \qquad f \in C_0(\Omega)$$

where

$$\tilde{U}^{(n)} = -\sum_{X\in\mathcal{F}:\ X-Y\subseteq\Lambda_n} \Phi_X^{\theta}.$$

Since μ is a unique Gibbs measure for $\mathcal{E}(\Phi)$, lemma 2.1 yields that there is a subsequence $\{m_j\}$ of $\{n_j\}$ such that

$$\mu f = \lim_{j\to\infty} \frac{\left\langle f \exp\{-U^{(m_j)}\}\right\rangle_{\Lambda_{m_j}}}{\left\langle \exp\{-U^{(m_j)}\}\right\rangle_{\Lambda_{m_j}}} \qquad f \in C_0(\Omega)$$

where

$$U^{(n)} = -\sum_{X\in\mathcal{F}:\ X-Y-Y\subseteq\Lambda_n} \Phi_X.$$

Now note that

$$\tilde{U}^{(n)} = -\sum_{X\in\mathcal{F}:\ X-Y\subseteq\Lambda_n} \Phi_X^{\theta} = -\sum_{\tilde{X}\in\mathcal{F}:\ \tilde{X}-Y-Y\subseteq\Lambda_n} \Phi_{\tilde{X}} \circ \xi_\theta = U^{(n)} \circ \xi_\theta.$$

Thus, for any $f \in C_0(\Omega)$, we have

$$\tilde{\mu} f \circ \xi_\theta = \lim_{j\to\infty} \frac{\left\langle (f \exp\{-U^{(m_j)}\}) \circ \xi_\theta\right\rangle_{\Lambda_{m_j}}}{\left\langle \exp\{-U^{(m_j)}\} \circ \xi_\theta\right\rangle_{\Lambda_{m_j}}} \qquad f \in C_0(\Omega).$$

Combining this with (4.1) and lemma 4.1 we obtain the desired conclusion. □

Proof of theorem 3.1. Let us observe that

$$\nabla_k(f \circ \xi_\theta) = \left(\sum_{j\in k+Y} a_{j-k} \nabla_j f\right) \circ \xi_\theta \qquad \text{for} \quad f \in C_0^\infty(\Omega) \quad k \in \mathbb{Z}^d.$$

Thus

$$\mathcal{D}(f \circ \xi_\theta) = \mathcal{K}_\theta(f) \circ \xi_\theta \qquad \text{for} \quad f \in C_0^\infty(\Omega) \quad k \in \mathbb{Z}^d.$$

Now assume that $\mu \in \mathcal{G}(\Phi)$ satisfies $\mathbf{SG}(\mathcal{K}_\theta)$. Let $\tilde{\mu} \in \mathcal{G}_{-Y}(\Phi^\theta)$. Then, by lemma 4.2 for any $f \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}\tilde{\mu}\big(f \circ \xi_\theta - \tilde{\mu} f \circ \xi_\theta\big)^2 = \mu\big(f - \mu f\big)^2 \leqslant C\mu\mathcal{K}_\theta(f) = C\tilde{\mu}\mathcal{K}_\theta(f) \circ \xi_\theta \\ \leqslant C\tilde{\mu}\mathcal{D}(f \circ \xi_\theta).\end{aligned}$$

Since $f \mapsto f \circ \xi_\theta$ is a bijection on $C_0^\infty(\Omega)$, $\tilde{\mu}$ satisfies $\mathbf{SG}(\mathcal{D})$. Assume now that $\tilde{\mu} \in \mathcal{G}_{-Y}(\Phi^\theta)$ satisfies $\mathbf{SG}(\mathcal{D})$. Then for all f we have

$$\mu\big(f - \mu f\big)^2 = \tilde{\mu}\big(f \circ \xi_\theta - \tilde{\mu} f \circ \xi_\theta\big)^2 \leqslant C\tilde{\mu}\mathcal{D}(f \circ \xi_\theta) \leqslant C\mu\mathcal{K}_\theta(f).$$

Thus μ satisfies $\mathbf{SG}(\mathcal{K}_\theta)$, and the proof of the first part of the theorem is completed. The same arguments can be applied in a proof of the second part concerning logarithmic Sobolev inequalities. □

5. Dobrushin–Shlosman mixing and logarithmic Sobolev inequalities

Definition 5.1. We say that the local specification $\mathcal{E}(\Phi)$ satisfies the *Dobrushin–Shlosman mixing condition* iff there is an $X \in \mathcal{F}$ with $0 \in X$, and a family of non-negative numbers $\alpha_{l,j}$ for $l \notin X$ and $j \in X$ such that

$$\beta = 1 - \frac{1}{|X|} \sum_{l \notin X, j \in X} \alpha_{l,j} > 0 \tag{5.1}$$

and for all $l \notin X$, $k \in \mathbb{Z}^d$, $f \in C_0^\infty(\Omega)$ and $Z \subseteq X$ one has

$$\|\partial_{l+k}\mathbb{E}_{k+Z} f - \mathbb{E}_{k+Z}\partial_{l+k} f\|_{\mathrm{u}} \leqslant \sum_{j \in X} \alpha_{l,j} \|\partial_{j+k} f\|_{\mathrm{u}}. \tag{5.2}$$

Remark 5.1. Note that if the potential family is shift-invariant, then it satisfies the Dobrushin–Shlosman condition iff (5.1) holds and if (5.2) is satisfied for $k = 0$.

Remark 5.2. The Dobrushin–Shlosman condition ensures the uniqueness of the Gibbs measure μ for $\mathcal{E}(\Phi)$ (see, e.g., [S]).

For further references we recall the following result of Stroock and Zegarlinski (see, e.g., [S, SZ1], or [SZ2]).

Theorem 5.1. *Assume that Φ is a C^2 potential of finite range, and that the local specification $\mathcal{E}(\Phi)$ satisfies the Dobrushin–Shlosman mixing condition. Then the unique Gibbs measure μ satisfies* $\mathbf{LS}(\mathcal{D})$.

As a direct consequence of theorems 3.1 and 5.1 we have;

Corollary 5.1. *Let Φ be a C^2 potential of finite range. If $\mathcal{E}(\Phi)$ and $\mathcal{E}(\Phi^\theta)$ satisfy the Dobrushin–Shlosman mixing condition, then the unique Gibbs measure $\mu \in \mathcal{G}(\Phi)$ satisfies* $\mathbf{LS}(\mathcal{K}_\theta)$.

In the next result we show that there always exists a high-temperature region where our conditions are satisfied.

Proposition 5.1 (Small potential case). *Let Φ be a C^2 potential of a finite range R. Assume that*

$$\begin{aligned} &\sup_{k \in \mathbb{Z}^d} \|U_{\{k+Y\}}\|_{\mathrm{u}} < \tfrac{1}{4} \log\left(1 + (R + \operatorname{diam} Y)^{-1}\right) \\ &\sup_{k \in \mathbb{Z}^d} \|U_k\|_{\mathrm{u}} < \tfrac{1}{4} \log\left(1 + R^{-1}\right). \end{aligned} \tag{5.3}$$

Then there is a unique Gibbs measure μ for $\mathcal{E}(\Phi)$, and μ satisfies $\mathbf{LS}(\mathcal{K}_\theta)$.

Proof. Note that the range of Φ^θ is less than or equal to $R + \operatorname{diam} Y$. According to corollary 5.1 it is enough to show that (5.3) implies that $\mathcal{E}(\Phi^\theta)$ and $\mathcal{E}(\Phi)$ satisfy the Dobrushin–Shlosman mixing condition with $X = \{0\}$. To do this we have to prove that there are positive constants α and $\tilde{\alpha}$ satisfying $(R + \operatorname{diam} Y)\alpha < 1$ and $R\tilde{\alpha} < 1$ such that

$$\|\partial_l \mathbb{E}_k^\theta f - \mathbb{E}_k^\theta \partial_l f\|_{\mathrm{u}} \leqslant \alpha \|\partial_k f\|_{\mathrm{u}} \qquad \text{for all } k \neq l \quad \text{and} \quad f \in C_0^\infty(\Omega) \tag{5.4}$$

and

$$\|\partial_l \mathbb{E}_k f - \mathbb{E}_k \partial_l f\|_{\mathrm{u}} \leqslant \tilde{\alpha} \|\partial_k f\|_{\mathrm{u}} \qquad \text{for all } k \neq l \quad \text{and} \quad f \in C_0^\infty(\Omega). \tag{5.5}$$

Let $k, l \in \mathbb{Z}^d$, $k \neq l$. Note that for all f and ω we have

$$\left[\partial_l, \mathbb{E}_k^\theta\right] f(\omega) = \left(\partial_l \mathbb{E}_k^\theta f - \mathbb{E}_k^\theta \partial_l f\right)(\omega)$$
$$= \int_\Omega \int_\Omega \left(\rho_k(x \bullet_l (y \bullet_k \omega)) - \rho_k(y \bullet_k \omega)\right) f(x \bullet_l (y \bullet_k \omega))\mu(\mathrm{d}x)\mu(\mathrm{d}y)$$

where the density ρ_k of $\mathbb{E}_k^\theta$ is given by

$$\rho_k(\omega) = \frac{\exp\{-U_k^\theta(\omega)\}}{\langle \exp\{-U_k^\theta\}(\omega)} \qquad \omega \in \Omega.$$

Using (4.1) we obtain

$$\rho_k(\omega) = \frac{\exp\{-U_{k+Y} \circ \xi_\theta(\omega)\}}{\langle \exp\{-U_{k+Y} \circ \xi_\theta\}(\omega)} \qquad \omega \in \Omega.$$

Since

$$\int_\Omega \rho_k(x \bullet_l (y \bullet_k \omega))\mu_0(\mathrm{d}y) = 1 = \int_\Omega \rho_k(y \bullet_k \omega)\mu_0(\mathrm{d}y)$$

we have

$$\left[\partial_l, \mathbb{E}_k^\theta\right] f(\omega) = -\int_\Omega \int_\Omega (\rho_k(x \bullet_l (y \bullet_k \omega)) - \rho_k(y \bullet_k \omega))(\partial_k f)(x \bullet_l (y \bullet_k \omega))\mu_0(\mathrm{d}x)\mu_0(\mathrm{d}y).$$

Thus (5.4) holds true with

$$\alpha = \sup_k \sup_{x,y,\omega \in \Omega} \left(\frac{\rho_k(x \bullet_l (y \bullet_k \omega))}{\rho_k(y \bullet_k \omega)} - 1\right) \leqslant \sup_k \sup_{\omega, v \in \Omega} \frac{\rho_k(\omega)}{\rho_k(v)} - 1$$
$$\leqslant \sup_k \exp\{4\|U_{k+Y}\|_{\mathrm{u}}\} - 1$$

having the desired property. In the same way one can show that (5.3) yields (5.5) with $R\tilde{\alpha} < 1$. $\square$

6. A class of infinite-volume stochastic dynamics

In this section we briefly describe the construction of an infinite-volume Markov semigroup corresponding to a general square of the field form $\mathcal{K}$. We consider a configuration space given by a product space $\Omega \equiv M^{\mathbb{Z}^d}$, where M is a smooth compact and connected Riemannian manifold. Let $\mathcal{W} \equiv \{W_i\}_{i \in \mathbb{Z}^d}$ be a collection of C^∞ vector fields defined as a following lift of the given smooth vector fields w_i on M:

$$W_i f(\omega) \equiv w_i f(\omega_i|\omega).$$

Given a finite set Y we define the following vector fields on Ω:

$$W_Y \equiv \sum_{j \in Y} W_j.$$

With this notation we introduce the following square of the field forms:

$$\mathcal{K}(f) \equiv \sum_{k \in \mathbb{Z}^d} (W_{k+Y} f)^2$$

with a domain including smooth cylinder functions $f \in C_0^\infty(\Omega)$. Given a local specification $\mathcal{E}(\Phi)$ corresponding to a smooth potential of finite range, we can now introduce the following elementary Markov operators on $C^2(\Omega)$:

$$\mathcal{L}_Y f \equiv W_Y^2 f + \beta_Y \cdot W_Y f$$

where we have set

$$\beta_Y \equiv \operatorname{div} W_Y + W_Y U_Y$$

with

$$\operatorname{div} W_Y \equiv \sum_{j \in Y} \operatorname{div}_j W_j$$

and $\operatorname{div}_j W_j$ is defined by the corresponding lift of $\operatorname{div} w_j$ on the manifold $\boldsymbol{M}$. With this notation one can see that

$$\mathbb{E}_Y (W_Y f)^2 = \mathbb{E}_Y (f(-\mathcal{L}_Y f)).$$

For later purposes we introduce the following free Markov generator:

$$\mathcal{L}^0 \equiv \sum_{k \in \mathbb{Z}^d} W_{k+Y}^2 f.$$

We note that $\mathcal{L}^0$ is local, that is for any $f \in C^2$ dependent only on ω_j, $j \in \Lambda_f$, one has

$$\mathcal{L}^0 f = \sum_{k \in \mathbb{Z}^d} W_{(k+Y)\cap\Lambda_f}^2 f$$

and therefore $\Lambda_{\mathcal{L}^0 f} \subset \Lambda_f$. This property allows us to easily define a Markov semigroup $P_t^0 \equiv \mathrm{e}^{t\mathcal{L}^0}$ on $C_0(\Omega)$. For any finite set $\Lambda \in \mathcal{F}$ we introduce a finite-volume generator

$$\mathcal{L}_\Lambda f \equiv \mathcal{L}^0 f + \sum_k \beta_{(k+Y)\cap\Lambda} \cdot W_{(k+Y)\cap\Lambda} f$$

with a convention that $\beta_\emptyset \equiv 0$. We note that $\mathcal{L}_\Lambda$ is again local and therefore it is easy to construct the corresponding Markov semigroup $P_t^{(\Lambda)} \equiv \mathrm{e}^{t\mathcal{L}_\Lambda}$ on $C_0(\Omega)$.

With the above assumptions and notation the following result is true.

Theorem 6.1. *Suppose that*

$$\sup_{k \in \mathbb{Z}^d, X \subset Y} \|\beta_{k+X}\|_{\mathrm{u}} < \infty$$

and

$$D \equiv \sup_{k \in \mathbb{Z}^d, Z, \Lambda \in \mathcal{F} : |Z| \leqslant |Y|} \|W_Z(\beta_{(k+Y)\cap\Lambda})\|_{\mathrm{u}} < \infty.$$

Then for any $f \in C_0^1(\Omega)$ the following limit exists:

$$P_t f \equiv \lim_{\Lambda \to \infty} P_t^{(\Lambda)} f$$

with the generator $\mathcal{L}$ satisfying

$$\mu(f(-\mathcal{L}f)) = \mathcal{K}(f).$$

Moreover, the following exponential approximation property is true: for any $A \in (0, \infty)$ there is $B \in (0, \infty)$ such that

$$\|P_t f - P_t^{(\Lambda)} f\|_{\mathrm{u}} \leqslant \mathrm{e}^{-At} C(f)$$

with some constant $C(f) \in (0, \infty)$ dependent only on f and the field $\mathcal{W}$, provided that

$$\operatorname{dist}(\Lambda_f, \mathbb{Z}^d \setminus \Lambda) \geqslant Bt.$$

Proof. For $\Lambda_1 \in \mathcal{F}$ and $\Lambda_2 \equiv \Lambda_1 \cup \{i\}$, we have

$$P_t^{(\Lambda_2)} f - P_t^{(\Lambda_1)} f = \int_0^t \mathrm{d}s \, \frac{\mathrm{d}}{\mathrm{d}s} P_{t-s}^{(\Lambda_1)} P_s^{(\Lambda_2)} f = \int_0^t \mathrm{d}s \, P_{t-s}^{(\Lambda_1)} (\mathcal{L}_{\Lambda_2} - \mathcal{L}_{\Lambda_1}) P_s^{(\Lambda_2)} f. \tag{6.1}$$

Next we note that

$$(\mathcal{L}_{\Lambda_2} - \mathcal{L}_{\Lambda_1}) F = \sum_{k:\mathrm{dist}(k+Y,i)\leqslant R} \left[\beta_{(k+Y)\cap\Lambda_2} W_{(k+Y)\cap\Lambda_2} - \beta_{(k+Y)\cap\Lambda_1} W_{(k+Y)\cap\Lambda_1}\right] \tag{6.2}$$

where R is the range of the interaction. Hence taking into the account that we consider Markov semigroups here, we obtain

$$\|P_t^{(\Lambda_2)} f - P_t^{(\Lambda_1)} f\|_{\mathrm{u}} \leqslant \sup_{k\in\mathbb{Z}^d, X\subset Y} \|\beta_{k+X}\|_{\mathrm{u}} \sum_{Z:\ \exists k\ \mathrm{dist}(k+Y,i)\leqslant R, Z\subset k+Y} \int_0^t \mathrm{d}s \, \|W_Z P_s^{(\Lambda_2)} f\|_{\mathrm{u}}. \tag{6.3}$$

Thus to complete the proof it is sufficient to obtain a bound for $\|W_Z P_s^{(\Lambda_2)} f\|_{\mathrm{u}}$ for $Z \subset k + Y$, $k \in \mathbb{Z}^d$. To this end we note that

$$W_Z P_s^{(\Lambda_2)} f = P_s^{(\Lambda_2)} W_Z f + \int_0^s \mathrm{d}\tau \, P_{s-\tau}^{(\Lambda_2)} [W_Z, \mathcal{L}_{\Lambda_2}] P_\tau^{(\Lambda_2)} f. \tag{6.4}$$

Noting that $[W_Z, \mathcal{L}^0] = 0$ we have

$$\begin{aligned}[] [W_Z, \mathcal{L}_{\Lambda_2}] &= \left[W_Z, \sum_k \beta_{(k+Y)\cap\Lambda_2} W_{(k+Y)\cap\Lambda_2}\right] \\ &= \sum_{k:\mathrm{dist}(Z,(k+Y)\cap\Lambda_2)\leqslant R} W_Z(\beta_{(k+Y)\cap\Lambda_2}) W_{(k+Y)\cap\Lambda_2}. \end{aligned} \tag{6.5}$$

From (6.4) and (6.5) we conclude that

$$\|W_Z P_s^{(\Lambda_2)} f\|_{\mathrm{u}} \leqslant \|W_Z f\|_{\mathrm{u}} + D \sum_{k:\mathrm{dist}(Z,(k+Y)\cap\Lambda_2)\leqslant R} \int_0^s \mathrm{d}\tau \, \|W_{(k+Y)\cap\Lambda_2} P_s^{(\Lambda_2)} f\|_{\mathrm{u}} \tag{6.6}$$

with

$$D \equiv \sup_{k\in\mathbb{Z}^d, Z, \Lambda\in\mathcal{F}:|Z|\leqslant|Y|} \|W_Z(\beta_{(k+Y)\cap\Lambda})\|_{\mathrm{u}}.$$

Given the inequality (6.6) the rest of the proof goes in a standard way (see, e.g., [GZ1]). □

7. Exponential decay to equilibrium for Kawasaki dynamics

Let the single spin space be given by S^1. We choose Y to be a set consisting of the origin and one of its nearest neighbours i_1 and $\boldsymbol{a} = \{-1, +1\}$. Then by theorem 3.1 a unique Gibbs measure μ_Φ related to a finite-range potential Φ satisfies **SG** or **LS** with the corresponding form

$$\bar{\mathcal{K}}(f) \equiv \sum_k |(\nabla_{k+i_1} - \nabla_k) f|^2$$

provided these inequalities remain true for a unique Gibbs measure μ_{Φ^θ} with the diagonal form. This naturally implies that **SG**, respectively, **LS**, is true for the form

$$\mathcal{K}(f) \equiv \sum_{j,k:|j-k|=1} |(\nabla_k - \nabla_j) f|^2$$

which is not smaller than $\bar{\mathcal{K}}$. As we have indicated in section 5 such a situation is true for any potential of finite range, provided the temperature of the system is sufficiently high (cf proposition 5.1). In particular, if $\mathbf{LS}(\mathcal{K})$ is satisfied, then the corresponding semigroup is hypercontractive. This together with the strong approximation property (theorem 6.1) allows one to apply the general strategy of Holley and Stroock (see, e.g., [SZ1]) to prove the uniform exponential decay to equilibrium. Thus we conclude with the following result.

Theorem 7.1. *Suppose for a finite-range potential Φ, the local specification $\mathcal{E}(\Phi^\theta)$ satisfies the mixing condition. Then the Kawasaki dynamics $P_t \equiv \mathrm{e}^{t\mathcal{L}}$ is strongly exponentially ergodic, that is for any function $f \in C_0^1(\Omega)$ we have*

$$\|P_t f - \mu_\Phi f\|_{\mathrm{u}} \leqslant C_\alpha \mathrm{e}^{-\alpha m t} \sum_k \|\nabla_k f\|_{\mathrm{u}}$$

with $m \equiv \mathrm{gap}_{\mathbb{L}_2(\mu)}(-\mathcal{L})$ and any $\alpha \in (0, 1)$ with a constant $C_\alpha \equiv C_\alpha(\Lambda_f)$ dependent only on Λ_f and the choice of α.

We stress that our mixing requirement involves the transformed potential. We note that the conditions are always satisfied in one dimension (as our transformation ξ^θ transforms finite-range potentials into finite-range potentials). Clearly, in higher dimensions the domain of strong mixing may depend on the potential (but in any case there always exists a non-trivial high-temperature region where the required mixing is true).

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References

[BZ1] Bertini L and Zegarlinski B 1999 Coercive inequalities for Kawasaki dynamics: the product case *Markov Process. Rel. Fields* **5** 125–62

[BZ2] Bertini L and Zegarlinski B 1999 Coercive inequalities for Gibbs measures *J. Funct. Anal.* **162** 257–86

[CM] Cancrini and Martinelli F On the spectral gap of Kawasaki dynamics under a mixing condition revisited mp_arc 5 *Preprint* 99-27

[De] Deuschel J D 1991 Algebraic $\mathbb{L}^2$ decay of attractive critical processes on the lattice *Ann. Probab.* **22** 264–83

[DS] Dobrushin R L and Shlosman S B 1985 Constructive criterion for the uniqueness of Gibbs field, completely analytical Gibbs fields, statistical physics and dynamical systems. Rigorous results *2nd Colloq. Workshop on Random Fields: Rigorous Results in Statistical Mechanics (Koszeg) (Progress in Physics vol 10)* ed J Fritz, A Jaffe and D Szasz (Boston, MA: Birkhäuser) pp 347–70, 371–403

[F] Faris W G 1979 The stochastic Heisenberg model *J. Funct. Anal.* **32** 342–52

[GZ1] Guionnet A and Zegarlinski B 1996 Decay to equilibrium in random spin systems on a lattice *Commun. Math. Phys.* **181** 703–32

[GZ2] Guionnet A and Zegarlinski B 1997 Decay to equilibrium in random spin systems on a lattice, II *J. Stat. Phys.* **86** 899–904

[HS1] Holley R and Stroock D W 1981 Diffusions on an infinite-dimensional torus *J. Funct. Anal.* **42** 29–63

[HS2] Holley R and Stroock D 1987 Logarithmic Sobolev inequalities and stochastic Ising models *J. Stat. Phys.* **46** 1159–94

[JLQY] Janvresse E, Landim C, Quastel J and Yau H T Relaxation to equilibrium of conservative dynamics I: zero range processes *Preprint*

[LY] Lu S L and Yau H T 1993 Spectral gap and logarithmic Sobolev inequality for Kawasaki dynamics and Glauber dynamics *Commun. Math. Phys.* **156** 399–433

[S] Stroock D W 1993 Logarithmic Sobolev inequalities for Gibbs states Dirichlet forms *Lectures Given at the 1st Session of the Centro Internazionale Matematico Estivo (CIME) (Varenna, 1992) (Lecture Notes in Mathematics vol 1563)* ed G Dell'Antonio and U Moscom (Berlin: Springer) pp 194–230

[SZ1] Stroock D W and Zegarlinski B 1992 Logarithmic Sobolev inequality for continuous spin system *J. Funct. Anal.* **104** 299–326

[SZ2] Stroock D W and Zegarlinski B 1992 The equivalence of the logarithmic Sobolev inequality and the Dobrushin–Shlosman mixing condition *Commun. Math. Phys.* **149** 175–93

[W] Wick W D 1981 Convergence to equilibrium of the stochastic Heisenberg model *Commun. Math. Phys.* **81** 361–77

[ZZ] Zhengping Zhang 1995 Generalized Kawasaki dynamics of the Heisenberg model *Phys. Rev.* E **51** 4155–8